Combined L1 / L2C Tracking Scheme for Weak Signal Environments

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BIOGRAPHY

Cyrille Gernot is a PhD. student in the Department of Geomatics Engineering at the University Of Calgary Schulich School of Engineering. He arrived in Calgary in early 2006 to perform a six month internship in the PLAN Group under the supervision of Dr. Gerard Lachapelle. This training period concluded 5 years of general engineering education with 2 years of preparation school and 3 years at Telecom INT and eventually ended up with a proposition to stay in Calgary as a MSc. student. After a few months in the PLAN Group, he finally transferred from MSc. to PhD. at the beginning of May 2007. He expects to complete his PhD. in September 2009.

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ABSTRACT

GPS receiver performances are strongly dependent on the user environment. For instance, attenuation of 20 to 30 dB or more can easily be reached indoor or in urban canyon environments. In order to overcome these limitations, one usually has to significantly increase the coherent integration time used. However, even though one was to use Assisted-GPS or the pilot channel proposed by new signals to eliminate the problem of data bit transitions, the use of long dwelling time remains limited by dynamics factors such as oscillator stability or user motion.

On the other hand, GPS modernization duplicates the number of signals available. Therefore, the coherent integration time required to increase the receiver sensitivity can significantly be reduced through the combination of signals transmitted from the same satellite.

In this paper, two methods combining L1 C/A and L2C signals energy at the tracking loop level are presented. One of the main difficulties when combining the signals transmitted at different frequencies in the tracking loop arises due to the ionosphere. Indeed, typical L1 / L2 tracking loop basically assumes that the Doppler frequency ratio between L1 and L2 is directly related to the theoretical scale factor f2 / f1 = 0.7792 and uses one signal to perform aiding on the other. However, the Doppler shift induced by the ionosphere is a function of the inverse transmitted frequency, thereby relating L1 and L2 Dopplers through the factor f1 / f2. In order to solve this problem, the output of the DLL and PLL discriminators are combined through a least-squares method estimating the TEC change and L2 code and phase error. In order to apply this method, specific considerations are given to the observation covariance matrix of the least squares method and an analysis of the PLL atan and DLL normalized early minus late envelope discriminators statistics is proposed.

INTRODUCTION

GPS satellites are roughly orbiting 20,000 km above the earth surface. As such, signals have to undergo a tremendous amount of free space loss before reaching the users. At the same time, signals have to go through ionosphere and troposphere. Whereas all these difficulties were taken into account during the design of the system and as such the -158.5 dBW of the legacy L1 C/A signal is easily acquired and tracked by receivers under open sky conditions, urban canyon and indoor environments add
other challenges limiting the scope of the GPS system (Figure 1).

![GPS signal path](image)

**Figure 1:** GPS signal path between satellite and indoor user

Indeed, attenuation up to 25 dB or more can easily be encountered under those adverse conditions. Common solutions aiming to overcome these difficulties include a significant increase in the coherent integration time leading to the well known data bit transitions problem (Watson et al. 2007). Indeed, as the L1 C/A signal carries the navigation message at 50 Hz, one cannot use coherent integration time longer than 20 ms without using Assisted-GPS (use of external information to perform data wipe-off) or performing a data bit estimation technique. However, both techniques remain limited either by the availability of external information or the user motion and local oscillator stability.

In order to overcome the data bit transition limitation and remove the need for assistance data or estimation techniques, the Department of Defense of the United States is currently modernizing the GPS system through the addition of two new signals namely L2C (for civilians) and L5 (for aviation). For instance, the L2C signal is now composed of a data channel carrying the navigation message as well as a pilot channel free of any data bits. The data channel is modulated by the so-called long ranging code (CL code) lasting 1.5 s and containing 767,250 chips clocked at 511.5 kHz. Note that, as L2 GPS already contained the military P(Y) code, the L2C signal is transmitted on a single bi-phase carrier and the CM and CL code are then time multiplexed at 1,023 MHz. Finally, even though transmitted 1.5 dB lower than legacy L1 C/A, L2C represents an extraordinary opportunity to enhance GPS performance as it has duplicated the number of signals available to civilians and also increased the total power available to users.

Taking advantage of this new signal, new acquisition and tracking algorithms have been developed during the last few years. For instance, Psiaki (2004) proposed a FFT-based CM / CL acquisition method for weak signal conditions; Lim et al (2006) developed a fast acquisition scheme based on L1 aiding L2 to perform the CM code phase and carrier Doppler estimations. Complete combined L1 / L2 acquisition taking advantage of the power duplication was proposed by Gernot et al (2007). Regarding tracking algorithms, combination of the L2C pilot and data channel and assessment of performance were discussed by Muthuraman et al (2007). Finally, other signal combinations across frequencies were considered by Ioannides et al (2007) for combined acquisition of GPS L1 and L5.

In this paper, two least-squares methods carrying the energy combination technique to the next stage by using both L1 C/A and L2C signals simultaneously in a combined tracking loop method to enhance performance under adverse conditions are presented. Note that, for the sake of the following discussion and considering the fact that no data bits are transmitted on the L2C signals yet, the L2C CM and CL code are simply considered as one code which is equivalent to adding the output of the CM and CL correlators. The first method proposed links L1 and L2 PLL discriminators outputs to each other through the estimation of the variation of the TEC encountered on the signal path between two instants and the phase error on L2. The second method proposes to increase the number of observation with the addition of the L1 and L2 DLL discriminators outputs to the measurements vector and includes the code error in its state vector in addition of the TEC variation and the phase error.

**IONOSPHERE EFFECTS**

When travelling through the ionosphere, the GPS signal is affected in two different ways. First of all, the code is delay by a factor proportional to the inverse of the square of the transmitted frequency $f$ and the total electron content (TEC expressed TEC unit) encountered on the signal path (Equation 1).
\[ \tau_c = \frac{40.3}{cf^2} \times 10^{16} \text{TEC} \quad (1) \]

\( \tau_c \) representing the code delay in second.

On the other hand, another effect known as phase advance affects the carrier frequency of the transmitted signal. In this case, the signal phase is advanced by the following factor in cycle (Equation 2):

\[ \phi_p = \frac{40.3}{cf} \times 10^{16} \text{TEC} \quad (2) \]
c representing the speed of light.

In order to illustrate these two effects, complex I and Q samples were collected under open sky conditions. Then the L1 C/A signal was used to track both L1 and L2 GPS signals, the principle being simply to track L1 and feeding the output parameters of the DLL and PLL tracking loop to L2 tracking. Figure 2 represents the equivalent PLL tracking loop used. A similar scheme could be drawn in term of DLL tracking.

![Figure 2: L1 PLL feeding L2 tracking](image)

Figure 3 and Figure 4 show the results obtained in term of code discriminator outputs code for L1 and L2 and real part of the prompt correlators for L1 and L2 respectively.

![Figure 3: L1 and L2 code discriminator outputs](image)

![Figure 4: output of the real part of the L1 and L2 prompt correlators](image)

The code delay created by the ionosphere can be easily seen in Figure 3 as the DLL discriminator output of the L2 signal shows an offset of about 0.02 chips. As such, in a common DLL scheme, this offset informs the DLL tracking loop that the local code used to track the signal is late compared to the incoming code and should be advanced by 0.02 chips.

On the other hand, the phase advance phenomenon expected should be observable in Figure 4 as a lower amplitude for the real part of the prompt correlator on L2 compared to L1. Indeed, the phase of the local carrier for L1 should be synchronized to the phase of the incoming carrier. However, due to the ionosphere induced phase difference between L1 and L2 incoming signal, the phase of the local L2 carrier which is identical to the phase of L1 should have an offset with its incoming signal. As such, the visible effect would be that the L2 incoming signal power should be shared between the real and complex part of the L2 prompt correlator. However, the observed phenomenon is a residual carrier frequency error for L2 signal. This last point is due to the fact that a simple phase shift would be observed only if the TEC encountered on the signal path would not change over time. This becomes untrue as the satellite motion results...
in a change of signal path through the ionosphere. Therefore if one was to assume the ionosphere to be a layer between the satellite and the user, the TEC encountered by the signal directly depends on the satellite elevation and change as the satellite is moving (Figure 5). This change induces a change in the phase advance observed and as such a frequency shift (Equation 3).

\[ \frac{d\varphi_p}{dt} = f_{DP} = \frac{40.3}{c^f} \times 10^{16} \frac{d\text{TEC}}{dt} \quad (3) \]

\( f_{DP} \) being the Doppler frequency induced by the ionosphere.

![Figure 5: Effect of satellite motion on signal path through ionosphere layer](image)

**TRACKING LOOP AIDING**

As shown in Figure 3 and Figure 4, tracking one of the two signals only and feeding the results of the tracking loop to the other leads to a bad condition tracking for the fed signal. Note that this effect would not be seen if one was to use a similar method for signals transmitted on the same frequency such as Galileo E1-B and E1-C.

In order to solve the ionosphere problem and be able to track both signal properly, one can either track them independently or use one to aid the other. In the following section, the possibility of using L1 C/A tracking loop to aid L2C is investigated but a similar analysis could be conducted on L2C aiding L1 C/A. The basic principle for the PLL is similar to the feeding technique but for the fact that L2 is actually tracked while using the scale Doppler frequency of L1 as aiding (Figure 6). A similar process is used regarding the DLL.

![Figure 6: L1 PLL aiding L2 PLL](image)

![Figure 7: L1 aiding L2 DLL discriminators outputs](image)

![Figure 8: L1 aiding L2, real part of prompt correlators outputs](image)
As shown through Figure 7, Figure 8 and Figure 9, performing aiding on L2 using the L1 signal permits to solve for the ionospheric phase shift and code delay observed previously. Indeed, the DLL discriminator of the L2 signal now converge toward zero and the real part of the prompt correlator remains constant. In order to show that the ionosphere phase advance was completely removed, the real and complex parts of L2 prompt correlator were plotted in Figure 9 where it can be observed that the signal power is concentrated on the real part only. This last point demonstrates that the phase of the incoming L2 signal is tracked.

Unfortunately, while being able to solve for the difficulties engendered by the ionosphere, the aiding technique does not actually take combined the two signals. Indeed, it only uses one to aid the other tracking. As such, the performance of the method is directly dependent on the signal offering the greatest power. A direct analogy of the aiding method would be the measurement of a distance using to different tape measures but instead of doing the mean of the obtained distance, one uses the measurements of first one to correct the measurements of the second one. In the case at hand, L1 is used to aid the L2 tracking loop but the measurements of L2 are never used to aid L1. An actual combination of the two signals would use both L1 and L2 measurements to obtain a better estimate of the desired quantities.

LEAST-SQUARES METHOD USING THE PHASE DISCRIMINATORS

The least-squares method used to combine L1 C/A and L2C GPS signals is based on the L1 and L2 phase discriminator outputs (Figure 10) which are linked together to estimate desired parameters which are then passed through the loop filter.

Before detailing the parameters estimated, it is important to note that the relation between phase and TEC is:

$$\phi = \phi + \frac{40.3}{c_f} 10^{16} TEC \quad (4)$$

$$\phi$$ being the total signal phase in cycle and $$\phi$$ representing the phase in cycle that would be measured if no ionosphere effect was present that is to say the phase variation change in range between the satellite and user.

As such, the total phase variation between two instants is:

$$\delta \phi = \delta \phi + \frac{40.3}{c_f} 10^{16} \delta TEC \quad (5)$$

Assuming that one has achieved phase lock on the signal the quantity represented in equation (5) can then be directly related to the average phase error between two coherent integration that is to say to the output of the phase discriminator $$\hat{\delta \phi}$$. Therefore if one was to use the atan discriminator due to its convenient auto-normalization properties, this last point can be summarized in equation (6).

$$\delta \phi = E(\hat{\delta \phi}) = E(\frac{Q_f}{I_f})/2\pi \quad (6)$$

Note that the atan discriminator has its output limited to the interval $$[-\pi/2 ; \pi/2]$$. However, the purpose of the discussion at hand is to keep the coherent integration time lower than 20 ms in order to avoid the need for data bit estimation or assistance data. As such, even if an error of 10 Hz in the Doppler frequency was made and a 50 vertical TEC was observed, the error created due to the Doppler frequency would be 1.25 rad and the change in phase due to the TEC would be about $$3.10^{-3}$$ rad. Therefore, the total phase error would still be included between $$[-\pi/2 ; \pi/2]$$.

Finally, by noticing that the average phase variation due to motion is directly related to the Doppler, one can write:
\[
\delta \phi_2 = \frac{f_1}{f_2} \delta \phi_1 \quad (7)
\]

At this point, the attentive reader would recognize the scale factor mentioned by Ioannides et al. (2007) and Gernot et al. (2007).

\[
SF = \frac{f_1}{f_2} \quad (8)
\]

Therefore by rewriting equations 5 to 8 for L1 and L2 signal, the following relation can be obtained and used as in least-squares method to estimate the phase change on L2 due to motion and the variation in TEC encountered on the signal path.

\[
\begin{bmatrix}
\delta \phi_1 \\
\delta \phi_2
\end{bmatrix} = \begin{bmatrix} SF & 10^{16} \frac{k}{f_1} \\
1 & 10^{16} \frac{k}{f_2}
\end{bmatrix} \begin{bmatrix}
\delta \phi_2 \\
\delta \text{TEC}
\end{bmatrix} \rightarrow Z = HX \quad (9)
\]

with \( k = \frac{40.3}{c} \).

Then these two estimates are passed through the loop filter which acts as a low-pass filter to reduce the noise corrupting the desired parameters. Once filtered, the obtained estimates are used to compute the appropriate correction for L1 and L2 local replica.

In order to use this method properly, the corresponding covariance matrix of the measurements \( C \) must be computed. Note that as L1 and L2 are transmitted on different frequency band, the noise corrupting the discriminator output on each frequency is independent.

\[
C_Z = \begin{bmatrix}
\sigma_{\delta \phi_1}^2 & 0 \\
0 & \sigma_{\delta \phi_2}^2
\end{bmatrix} \quad (10)
\]

However, computing the variance of the \textit{atan} discriminator proves to be challenging. Another approach proposed is to compute the variance of the simpler \( I.Q \) discriminator and compared it to the variance of the \textit{atan} discriminator obtained through Monte-Carlo simulation.

As shown in appendix A, the expected value and variance of the \( I.Q \) discriminator can be expressed as:

\[
E(\delta \phi) = E(I_p Q_p) = E(\text{atan}\left(\frac{Q_p}{I_p}\right)) = \delta \phi
\]

\[
\text{var}(\delta \phi) = \sigma_{\delta \phi}^2 = \sigma_N^2 + \sigma_N^4
\]

with \( \sigma_N^2 = \frac{1}{2C/T} \) and \( T \) being the coherent integration time.

Once the theoretical value of the variance of the \( I.Q \) computed, a Monte-Carlo simulation was conducted and the variances of the \( I.Q \) and \textit{atan} discriminator were obtained and compared to the theory as a function of the \( C/N_0 \). The results shown in Figure 11 demonstrate that the theoretical variance for the \( I.Q \) discriminator matched perfectly the Monte-Carlo simulation. Moreover, Figure 12 shows the difference between the theoretical variance of the \( I.Q \) discriminator and the variance of the \textit{atan} discriminator and illustrates that for \( C/N_0 \) superior to 20 dB-Hz, the approximation between \( I.Q \) and \textit{atan} discriminator variances hold.

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**Figure 11:** variance of the \( I.Q \) and \textit{atan} discriminator as a function of the \( C/N_0 \)

**Figure 12:** Difference between theoretical variance of \( I.Q \) and Monte-Carlo variance of \textit{atan} discriminator as a function of the \( C/N_0 \)

Finally, it is important to note that the least-squares method becomes optimal for symmetric noise distribution. Moreover, if the noise corrupting the
measurements vector $Z$ can be considered as Gaussian, the resulting noise on the state vector is Gaussian and has for covariance $C_X$.

$$C_X = \left( H^T C_Z^{-1} H \right)^{-1}$$

As such, an approximation of the noise corrupting the $\text{atan}$ discriminator by a Gaussian distribution of mean $\delta \phi$ and variance $\sigma^2_{\delta \phi}$ is performed. In order to verify the validity of such an approximation, the cumulative distribution functions (CDFs) of the $\text{atan}$ discriminator values obtained through the Monte-Carlo simulation are plotted for different values of $C/N_0$ and compared to the corresponding Gaussian noise CDFs (Figure 13).

Finally, Figure 14 displays the difference between the CDFs obtained from the Gaussian approximation and the CDFs obtained through Monte-Carlo for different value of $C/N_0$. The obtained results show that the approximation of the noise corrupting the $\text{atan}$ discriminator by a Gaussian distribution hold for $C/N_0$ values superior to 20 dB-Hz.

**LEAST-SQUARES METHOD USING BOTH PHASE AND CODE DISCRIMINATORS**

Where the previous method is acceptable, it is limited by the fact that the number of elements in the measurements vector is equal to the number of elements in the state vector. As such, better estimate could be obtained if one was able to increase the measurements vector length.

One solution to do so is to use the L1 and L2 code discriminators outputs in addition to the L1 and L2 phase discriminators outputs. Indeed, whereas the phase discriminator can be related to the phase error due to motion and the TEC variation so can be the code discriminator.

First of all, one can expressed the relation between the code delay, the range and the ionosphere variations:

$$\delta \tau = \delta l + \frac{c}{f} \delta \phi - \frac{40.3}{\lambda_c f^2} 10^{16} \delta \text{TEC} \quad (11)$$

$\delta \tau$ being the total change in code delay between two instant in chip, $\delta l$ (chip) the change in code delay due to motion only minus the change in phase due to motion (as such, it should converge toward zero), $f_c$ the code frequency and $\lambda_c$ the code wavelength.

Similarly than for the phase variation, the code variation can be related to the code discriminator as long as code lock was achieved. Therefore, using the popular normalized early minus late envelope; the following equations can be written:
\[ \delta\hat{\tau} = (1 - \Delta_{EL}) \frac{\sqrt{I_E^2 + Q_E^2} - \sqrt{I_L^2 + Q_L^2}}{\sqrt{I_E^2 + Q_E^2} + \sqrt{I_L^2 + Q_L^2}} \] (chip)

\( \Delta_{EL} \) being the distance between the early and late discriminator in chip also referred as early-late spacing.

As such, \( E(\delta\hat{\tau}) = \delta\tau \).

Therefore using equation 11 and equation 9, a new model making use of both phase and code discriminators can be derived. Note that as the code frequency is identical on L1 and L2, the code delays differ only by the ionosphere effect that is to say that no scale factor is required.

\[
\begin{bmatrix}
\hat{\phi}_1
\hat{\phi}_2
\hat{\delta}\tau_1
\hat{\delta}\tau_2
\end{bmatrix} =
\begin{bmatrix}
SF & 0 & 10^{16} & k \\
0 & 0 & 10^{16} & f_2 \\
\lambda_c & 1 & -10^{16} & 40.3 \\
\lambda_c & 1 & -10^{16} & 40.3 \\
\end{bmatrix}
\begin{bmatrix}
\hat{\phi}_2 \\
\hat{\delta}l_2 \\
\delta\text{TEC}
\end{bmatrix}
\]

However, the covariance matrix \( C_2 \) of the measurements become more complicated as one now needs to compute the variance of the code discriminator as well as the covariance between the code and phase discriminator in addition to the phase discriminator variance. Note that any covariance between L1 and L2 discriminator remains null as the noise on L1 is independent of the noise on L2.

\[
C_2 =
\begin{bmatrix}
\sigma_{\hat{\phi}_1}^2 & 0 & \text{cov}(\hat{\phi}_1, \hat{\delta}\tau_1) & 0 \\
0 & \sigma_{\hat{\phi}_2}^2 & 0 & \text{cov}(\hat{\phi}_2, \hat{\delta}\tau_1) \\
\text{cov}(\hat{\phi}_1, \hat{\delta}\tau_1) & 0 & \sigma_{\hat{\delta}\tau_1}^2 & 0 \\
0 & \text{cov}(\hat{\phi}_2, \hat{\delta}\tau_1) & 0 & \sigma_{\hat{\delta}\tau_1}^2
\end{bmatrix}
\]

Regarding the code discriminator variance, a similar approach than for the computation of the phase discriminator variance is taken. An approximation is made that the normalized early minus late discriminator has the same properties than the early minus late power discriminator. Then the equivalence is verified through Monte-Carlo simulation. Recall that the early minus late power discriminator is expressed as:

\[
E \left[ \frac{1}{2(2 - \Delta_{EL})} \left( I_E^2 + Q_E^2 - (I_L^2 + Q_L^2) \right) \right]
\]

Through the computation detailed in appendix B, it can be shown that:

\[
E \left[ \frac{1}{2(2 - \Delta_{EL})} \left( I_E^2 + Q_E^2 - (I_L^2 + Q_L^2) \right) \right] = E(\delta\hat{\tau}) = \delta\tau
\]

and

\[
\text{var}(\delta\hat{\tau}) = \sigma_{\delta\hat{\tau}}^2
\]

\[
= \sigma_w^2 \left[ 1/2 + \frac{2\delta\tau^2}{(2 - \Delta_{EL})^2} \right] - \left( 1 - \Delta_{EL} \right) \left[ 1/2 - \frac{2\delta\tau^2}{(2 - \Delta_{EL})^2} \right]
+ 2\sigma_w^4 \frac{\Delta_{EL}}{2 - \Delta_{EL}}
\]

By assuming that lock was achieved and \( \Delta_{EL} = 0.1 \) such that \( \delta\tau \) is small, a simplified expression can be obtained assuming \( \delta\tau \approx 0 \).

\[
\text{var}(\delta\hat{\tau}) = \sigma_{\delta\hat{\tau}}^2 = \sigma_w^2 \Delta_{EL} + 2\sigma_w^4 \frac{\Delta_{EL}}{2 - \Delta_{EL}}
\] (12)

In order to verify the aforementioned assumptions, a Monte-Carlo simulation was conducted to obtained the variances of the early minus late power and normalized early minus late envelope discriminators and compared them to the theoretical value computed. Note whereas the code delay introduced in the simulation is 0.05 chip and the early late spacing is 0.1 chip, the formula used for the theoretical variance is equation 12 where the code delay is assumed null. As shown in Figure 15, even though the zero code delay approximation was made in the computation of the theoretical variance, the discrepancy with the Monte-Carlo simulation variance of the early minus late power discriminator is negligible.

On the other hand, Figure 16 shows the difference between the theoretical variance of the early minus late power discriminator and the variance of the normalized early minus late envelope discriminator obtained through Monte-Carlo simulation and illustrates that for \( C/N_0 \) superior to 20 dB-Hz, the approximation between the two discriminators hold.
Regarding the covariance of the phase \( \text{atan} \) discriminator and the normalized early minus late envelope discriminator, the computation of a specific theoretical value proves to be especially difficult as the approximation of these discriminators by the \( I.Q \) and early minus late power discriminators does not hold for the covariance. This last point is shown in Figure 17 where the covariance of both pairs of discriminators was determined through Monte-Carlo simulation. However, the inability to determine a theoretical value is not of great concern as the Monte-Carlo simulation also shows that the covariance of the \( \text{atan} \) and normalized early minus late envelope discriminators is negligible for \( C/N_0 \) superior to 20 dB-Hz.

Finally, as the proposed method remains a least-squares technique, the approximation of Gaussian noise while not necessary is convenient in term of analysis of the noise corrupting the desired estimates. As such, a similar approximation than for the phase discriminator was conducted on the normalized early minus late envelope discriminator that is to say that the noise is assumed to have a Gaussian distribution of mean \( \delta \tau \) and variance \( \sigma_{\delta \tau}^2 \). Then, the CDFs were computed for different values of \( C/N_0 \) for both the Gaussian noise approximation and noise obtained through Monte-Carlo simulation. Results are displayed in Figure 18 and Figure 19 and show that the Gaussian noise approximation hold for \( C/N_0 \) superior to 20 dB-Hz.
CONCLUSION AND FUTURE WORK

In this paper, the possibility of combining signals transmitted on different frequency was discussed. The main problem in inter-frequency combination lays in the frequency dependent effects induced by the ionosphere and resulting in an additional code delay and phase advanced different for each signal. As such, it has been shown that using one signal only to track both L1 C/A and L2C is not possible as it results as a residual Doppler frequency error and a bad synchronization of the local code. In order to solve these difficulties, one can either track each signal independently or use one signal to aid the other. However, neither of these solutions actually combined the signals as they do not make use of both signals to obtain better estimate of the desired parameters and as such, the tracking performance are limited to the signal of greatest power. Therefore, two methods combining the outputs of the phase and code discriminators through a least-squares technique aiming to estimate the TEC variation on the signal path were proposed. Moreover, as these methods used the outputs of the discriminators, a complete derivation of the statistical properties of the I/Q and early minus late power discriminators was done. Then, the obtained mean and variance were used to approximate the popular auto-normalizing discriminator namely $\text{atan}$ and normalized early minus late envelope. Finally, the fact that one can approximate the noise corrupting the discriminators as Gaussian was demonstrated. Future work includes the characterization of the loop filter and the development of a Kalman filter based tracking using a similar measurements model.

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REFERENCES


APPENDIX A: DERIVATION OF THE I/Q DISCRIMINATOR EXPECTED VALUE AND VARIANCE

From Van Dierendonck (1996), the real and complex part of the prompt correlator can be expressed as (Equation 1a and 2a):
\[ I_P = \text{sinc}(\pi \Delta f T)D \sqrt{\frac{2}{N_0}} \frac{C}{T} R(\tau) \cos(\delta \phi) + n_i \]  
(1a)

\[ Q_P = \text{sinc}(\pi \Delta f T)D \sqrt{\frac{2}{N_0}} \frac{C}{T} R(\tau) \sin(\delta \phi) + n_Q \]  
(2a)

\( \Delta f \) being the frequency error, \( T \) the coherent integration time, \( D \) the data bit, \( R \) the auto-correlation function, \( \tau \) the code error, \( \delta \phi \) the phase error.

\( n_{i/q} \) represents the Gaussian noise corrupting the real and complex part of the correlator respectively and has the following properties:

\[ \text{var}(n_i) = \text{var}(n_Q) = 1 \]
\[ E(n_i) = E(n_Q) = E(n_in_Q) = 0 \]

Note that the I and Q part of the noise are not independent but can be considered uncorrelated due to the carrier removal process which multiplied it by the \( \sin \) and \( \cos \) which are orthogonal functions.

As the final purpose is to use the statistical properties of the I,Q discriminator as an approximation of the auto-normalizing \( \text{atan} \) discriminator, one has to normalize equations 1a and 2a by dividing by:

\[ \sqrt{\frac{2}{N_0}} \frac{C}{T} \]

Then, assuming that the Doppler error \( \Delta f \) and code error \( \tau \) are small and that one integrates over one data bit only, equation 1a and 2a can be rewritten (after normalization ) as:

\[ I_P = \cos(\delta \phi) + N_i \]  
(3a)

\[ Q_P = \sin(\delta \phi) + N_Q \]  
(4a)

with \( N_i = \frac{n_i}{\sqrt{\frac{2}{N_0}} \frac{C}{T}} \) and \( N_Q = \frac{n_Q}{\sqrt{\frac{2}{N_0}} \frac{C}{T}} \).

As such, the statistics of \( N_i \) and \( N_Q \) are:

\[ \text{var}(N_i) = \text{var}(N_Q) = \sigma^2 = \frac{1}{2 \frac{C}{N_0}} \]

and \( E(N_i) = E(N_Q) = E(N_iN_Q) = 0 \).

Finally, using equations 3a and 4a, one can access the statistical properties of the \( I,Q \) discriminator.

Calling \( \hat{\delta \phi} = I_P Q_P \), the expected value is:

\[ E(\hat{\delta \phi}) = E(I_P Q_P) \]
\[ = E[(\cos(\delta \phi) + N_i)(\sin(\delta \phi) + N_Q)] \]
\[ = E[\cos(\delta \phi)\sin(\delta \phi)] \]
\[ = \frac{1}{2} \sin(2\delta \phi) \]

Therefore, for small values of the phase error, the expected value reduces to the actual phase error:

\[ E(\hat{\delta \phi}) = E(I_P Q_P) = \delta \phi \]

Regarding the computation of the discriminator variance, the following formula is derived:

\[ \sigma^2 = E(\hat{\delta \phi}^2) - E(\hat{\delta \phi})^2 \]
\[ = E(I_P^2 Q_P^2) - E(I_P Q_P)^2 \]
\[ = E[\cos(\delta \phi)\sin(\delta \phi)] + 2E[\cos(\delta \phi)^2\sin(\delta \phi)]E[N_Q] \]
\[ + E[\cos(\delta \phi)^2]E[N_Q^2] + 2E[\cos(\delta \phi)\sin(\delta \phi)^2]E[N_i] \]
\[ + 4E[\cos(\delta \phi)\sin(\delta \phi)]E[N_i N_Q] \]
\[ + 2E[\sin(\delta \phi)^2]E[N_i^2 N_Q^2] + E[N_i^2 N_Q^2] \]
\[ + 2E[\sin(\delta \phi)^2]E[N_i^2 N_Q^2] + E[N_i^2 N_Q^2] \]
\[ - E[\cos(\delta \phi)\sin(\delta \phi)]^2 \]

Remembering the noise properties, this expression reduces to:

\[ \sigma^2 = E[\cos(\delta \phi)^2]E[N_Q^2] + E[\sin(\delta \phi)^2]E[N_i^2] \]
\[ + E[N_i^2 N_Q^2] \]

That is: \( \sigma^2 = \sigma_n^2 + \sigma_n^4 \)
APPENDIX B: DERIVATION OF THE EARLY MINUS LATE POWER DISCRIMINATOR EXPECTED VALUE AND VARIANCE

From equation 1a and 2a and by applying the normalization factor, the output of the early and late correlator can be expressed as (assuming small frequency error and integration period smaller or equal to data bit length):

\[ I_E = R\left(\tau - \frac{\Delta}{2}\right)\cos(\delta \phi) + N_{IE} \]  
\[ Q_E = R\left(\tau - \frac{\Delta}{2}\right)\sin(\delta \phi) + N_{QE} \]  
\[ I_L = R\left(\tau + \frac{\Delta}{2}\right)\cos(\delta \phi) + N_{IL} \]  
\[ Q_L = R\left(\tau + \frac{\Delta}{2}\right)\sin(\delta \phi) + N_{QL} \]

\( \Delta \) being the correlator spacing that is the number of chip between the early and late correlators.

However, before one begins the actual derivation of the early minus late power statistical properties, it is important to note that the noise corrupting the early and late discriminator is not completely uncorrelated. Indeed, even if one were to assume that the auto-correlation properties of the PRN code were perfect (hypothesis that is adopted herein), the output noise of the early and late correlators would be uncorrelated only if the correlator spacing \( \Delta \) was more or equal to 1 chip. As such, if \( \Delta \) is smaller than 1 as it is the case for narrow correlator, the noise corrupting the early and late correlators can then be divided in two parts:

\[ N_{IE} = W_{IEL} + w_{IE} \]  
\[ N_{IL} = W_{IEL} + w_{IL} \]  
\[ N_{QE} = W_{QEL} + w_{QE} \]  
\[ N_{QL} = W_{QEL} + w_{QL} \]

Note that the noise on the I and Q channels is uncorrelated and that:

\[ E(W_{IEL}) = E(w_{IE}) = E(w_{IL}) = 0 \]
\[ E(W_{QEL}) = E(w_{QE}) = E(w_{QL}) = 0 \]
\[ E(w_{IE}w_{IL}) = E(w_{QE}w_{QL}) = 0 \]
\[ E(W_{IEL}^2) = E(W_{QEL}^2) = (1 - \Delta)\sigma_N^2 \]
\[ E(w_{IE}^2) = E(w_{IL}^2) = \Delta\sigma_N^2 \]
\[ E(w_{QE}^2) = E(w_{QL}^2) = \Delta\sigma_N^2 \]

The definition proposed herein for the early minus late power discriminator is the following:

\[ \hat{\tau} = \frac{1}{2(2 - \Delta)}\left[I_E^2 + Q_E^2 - (I_L^2 + Q_L^2)\right] \]

For sake of simplicity, one also defines:

\[ \bar{I}_E = R\left(\tau - \frac{\Delta}{2}\right)\cos(\delta \phi) \]
\[ \bar{Q}_E = R\left(\tau - \frac{\Delta}{2}\right)\sin(\delta \phi) \]
\[ \bar{I}_L = R\left(\tau + \frac{\Delta}{2}\right)\cos(\delta \phi) \]
\[ \bar{Q}_L = R\left(\tau + \frac{\Delta}{2}\right)\sin(\delta \phi) \]
Using the previous definition and equation 1b, 2b, 3b and 4b, one can compute the expected value of $\hat{\tau}$:

$$E[I_E^2 + Q_E^2 - (I_L^2 + Q_L^2)] = I_E^2 + Q_E^2 - (I_L^2 + Q_L^2) + E(N_{IE}^2) + E(N_{QE}^2) - E(N_{IL}^2) - E(N_{QL}^2)$$

However,

$$E(N_{IE}^2) = E(N_{QE}^2) = E(N_{IL}^2) = E(N_{QL}^2)$$

So,

$$E[I_E^2 + Q_E^2 - (I_L^2 + Q_L^2)] = \left[R(\tau - \frac{A}{2})\right]^2 - \left[R(\tau + \frac{A}{2})\right]^2 = 2\tau(2 - \Delta)$$

Therefore,

$$E(\hat{\tau}) = \tau$$

On the other hand, the variance of the discriminator can be expressed as:

$$\sigma_\tau^2 = \left(\frac{1}{2(2 - \Delta)}\right)^2 [A - B]$$

with

$$A = E[I_E^2 + Q_E^2 - (I_L^2 + Q_L^2)]$$

and

$$B = E[I_E^2 + Q_E^2 - (I_L^2 + Q_L^2)]^2$$

Then, developing the expression of $A$:


Which can be expressed as:

$$A = A_1 - 2A_2 + A_3$$

with

$$A_1 = E[I_E^2 + Q_E^2]^2$$

$$A_2 = E[I_E^2 + Q_E^2](I_L^2 + Q_L^2)]$$

$$A_3 = E[I_L^2 + Q_L^2]^2$$

Similarly,

$$A_1 = E[I_E^2 + N_{IE}^2]^4 + 2E[I_E^2 + N_{IE}^2]Q_E + N_QE^2]$$

$$A_2 = E[I_E^2 + N_{IE}^2]^2(Q_E + N_QE^2)]$$

$$A_3 = E[Q_E + N_QE^2]^4$$

Expressing $A_{11}$:

$$A_{11} = I_E^4 + 6I_E^2E(N_{IE}^2) + 4I_EE(N_{IE}^3) + E(N_{IE}^4)$$

Note that $N_{IE}$ is Gaussian zero-mean and of variance $\sigma_N^2$. As such, it has for probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-\frac{x^2}{2\sigma_N^2}}.$$}

Therefore,

$$E(N_{IE}^3) = \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2\sigma_N^2}} dx$$

$$= \int_{-\infty}^{0} x^3 e^{-\frac{x^2}{2\sigma_N^2}} dx + \int_{0}^{\infty} x^3 e^{-\frac{x^2}{2\sigma_N^2}} dx$$

$$= \int_{0}^{\infty} x^3 e^{-\frac{x^2}{2\sigma_N^2}} dx + \int_{0}^{\infty} x^3 e^{-\frac{x^2}{2\sigma_N^2}} dx$$

$$= 0$$

Similarly, one can show that:
\[ E(N_{IE}^4) = 2 \int_0^{+\infty} \frac{x^4}{\sqrt{2\pi}\sigma_N^2} e^{-\frac{x^2}{2\sigma_N^2}} dx \]

However,

\[
\int_0^{+\infty} x^n e^{-ax^2} dx = \left\{ \begin{array}{ll}
\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2\pi}} \frac{2^{\frac{n+1}{2}}}{a^{\frac{n+1}{2}}} & n > -1, a > 0 \\
\frac{2(a-k)!}{k!} \frac{2\pi}{a^{\frac{n+1}{2}}} & n = 2k, k \text{ integer}, a > 0 \\
\frac{2(a-k)!}{k!} \frac{2\pi}{a^{\frac{n+1}{2}}} & n = 2k+1, k \text{ integer}, a > 0
\end{array} \right.
\]

Note that \( \Gamma \) represents the Gamma function and the operator \( !! \) is the double factorial.

Using this integral with the problem at hand, one can show that:

\[ E(N_{IE}^4) = 3\sigma_N^4. \]

Therefore, the expression of \( A_{11} \) reduces to:

\[ A_{11} = \bar{T}_E^2 + 6\bar{T}_E^2\sigma_N^2 + 3\sigma_N^4. \]

Similarly, it can be shown that:

\[ A_{12} = \bar{T}_E^2\bar{Q}_E^2 + \bar{T}_E^2\sigma_N^2 + \bar{Q}_E^2\sigma_N^2 + \sigma_N^4 \]

and

\[ A_{13} = \bar{Q}_E^2 + 6\bar{Q}_E^2\sigma_N^2 + 3\sigma_N^4 \]

which reduces the expression of \( A_1 \) to:

\[ A_1 = (\bar{T}_E^2 + \bar{Q}_E^2)^2 + 8\sigma_N^2 (\bar{T}_E^2 + \bar{Q}_E^2) + 8\sigma_N^4. \]

Moreover, by noticing the strong similarities between the expression of \( A_1 \) and \( A_3 \), one can readily determine that:

\[ A_3 = (\bar{T}_L^2 + \bar{Q}_L^2)^2 + 8\sigma_N^2 (\bar{T}_L^2 + \bar{Q}_L^2) + 8\sigma_N^4. \]

From this point forward, only \( A_2 \) is required to finalize the computation of \( A \).

\[ A_2 = E\left[ (I_E^2 + Q_E^2)(I_L^2 + Q_L^2) \right] \]
\[ = E(I_E^2I_L^2) + E(I_E^2Q_L^2) + E(Q_E^2I_L^2) + E(Q_E^2Q_L^2) \]
\[ = A_{21} + A_{22} + A_{23} + A_{24} \]

with

\[ A_{21} = E(I_E^2I_L^2) \]
\[ A_{22} = E(I_E^2Q_L^2) \]
\[ A_{23} = E(Q_E^2I_L^2) \]
\[ A_{24} = E(Q_E^2Q_L^2) \]

Deriving \( A_{2j} \) gives:

\[ A_{21} = \bar{T}_E^2\bar{T}_L^2 + \bar{T}_E^2E(N_{IL}^2) + 4\bar{T}_E\bar{T}_L E(N_{IE}N_{IL}) + 2\bar{T}_E E(N_{IE}N_{IL}^2) + \bar{T}_L^2 E(N_{IE}^2N_{IL}) + E(N_{IE}^2N_{IL}^2) \]

However, by using the interpretation of the noise corrupting the early and late correlators given earlier, one can show that:

\[ E(N_{IL}^2) = E(N_{IE}^2) = \sigma_N^2 \]
\[ E(N_{IE}N_{IL}^2) = (N_{IE}^2N_{IL}) = 0 \]
\[ E(N_{IE}N_{IL}) = (1-\Delta)\sigma_N^2 \]
\[ E(N_{IE}^2N_{IL}^2) = \left[ 2(1-\Delta)^2 + 1 \right] \sigma_N^4 \]

Therefore,

\[ A_{21} = \bar{T}_E^2\bar{T}_L^2 + \bar{T}_E^2\sigma_N^2 + 4\bar{T}_E\bar{T}_L(1-\Delta)\sigma_N^2 + \bar{T}_L^2\sigma_N^2 + \left[ 2(1-\Delta)^2 + 1 \right] \sigma_N^4 \]

The expressions of \( A_{22}, A_{23} \) and \( A_{24} \) can be similarly determined:

\[ A_{22} = \bar{T}_E^2\bar{Q}_L^2 + \bar{T}_E^2\sigma_N^2 + \bar{Q}_L^2\sigma_N^2 + \sigma_N^4 \]
\[ A_{23} = \bar{Q}_E^2\bar{T}_L^2 + \bar{Q}_E^2\sigma_N^2 + \bar{T}_L^2\sigma_N^2 + \sigma_N^4 \]
\[ A_{24} = \bar{Q}_E^2\bar{Q}_L^2 + \bar{Q}_E^2\sigma_N^2 + 4\bar{Q}_E\bar{Q}_L(1-\Delta)\sigma_N^2 + \bar{Q}_L^2\sigma_N^2 + \left[ 2(1-\Delta)^2 + 1 \right] \sigma_N^4 \]

Then using \( A_{21}, A_{22}, A_{23} \) and \( A_{24} \), \( A_2 \) can be expressed as:
\[ A_2 = (\vec{I}_E^2 + \overline{Q}_E^2)(\vec{I}_L^2 + \overline{Q}_L^2) \]
\[ + 2(\vec{I}_E^2 + \overline{Q}_E^2 + \vec{I}_L^2 + \overline{Q}_L^2)\sigma_N^2 \]
\[ + 4(\vec{I}_E \vec{I}_L + \overline{Q}_E \overline{Q}_L)(1 - \Delta)\sigma_N^2 + 2[2(1 - \Delta)^2 + 2]\sigma_N^4 \]

Finally, combining \( A_1, A_2 \) and \( A_3 \) yields the expression of \( A \):
\[
A = A_1 - 2A_2 + A_3 \\
= \left[ (\vec{I}_E^2 + \overline{Q}_E^2) - (\vec{I}_L^2 + \overline{Q}_L^2) \right]^2 \\
+ 4(\vec{I}_E^2 + \overline{Q}_E^2 + \vec{I}_L^2 + \overline{Q}_L^2)\sigma_N^2 \\
- 8(\vec{I}_E \vec{I}_L + \overline{Q}_E \overline{Q}_L)(1 - \Delta)\sigma_N^2 \\
+ 8\Delta(2 - \Delta)\sigma_N^2
\]

Note that \( B \) is equal to:
\[
B = \left[ (\vec{I}_E^2 + \overline{Q}_E^2) - (\vec{I}_L^2 + \overline{Q}_L^2) \right]^2
\]

Therefore,
\[
A - B = 4(\vec{I}_E^2 + \overline{Q}_E^2 + \vec{I}_L^2 + \overline{Q}_L^2)\sigma_N^2 \\
- 8(\vec{I}_E \vec{I}_L + \overline{Q}_E \overline{Q}_L)(1 - \Delta)\sigma_N^2 \\
+ 8\Delta(2 - \Delta)\sigma_N^2
\]

and
\[
\sigma_z^2 = \left( \frac{1}{2(2 - \Delta)} \right)^2 [A - B]
\]

can finally be expressed as:
\[
\sigma_z^2 = \sigma_N^2 \left[ 1/2 + \frac{2\tau^2}{(2 - \Delta)^2} - (1 - \Delta)\left( \frac{1/2 - 2\tau^2}{(2 - \Delta)^2} \right) \right] \\
+ 2\sigma_N^4 \frac{\Delta}{2 - \Delta}
\]